

**my input: are you familiar with the Black Scholes model?**

Yes, I am familiar with the Black-Scholes model. The Black-Scholes model, introduced by Fischer Black and Myron Scholes in 1973 (with key contributions by Robert Merton), is a groundbreaking approach for valuing options and other derivative securities. Its primary contribution is a formula that provides a theoretical price for European-style options based on various input parameters, assuming that the price of the underlying asset follows a geometric Brownian motion (GBM).

The Black-Scholes model for the price of a European call option  $C$  on an underlying asset with current price  $S_0$ , strike price  $K$ , time to expiration  $T$ , risk-free interest rate  $r$ , and volatility  $\sigma$  is given by:

$$C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

where:

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T}$$

Here: -  $\Phi(d)$  denotes the cumulative distribution function of the standard normal distribution. -  $S_0$  is the current price of the asset. -  $K$  is the option's strike price. -  $T$  is the time to expiration. -  $r$  is the risk-free interest rate. -  $\sigma$  is the volatility of the asset's returns.

Assumptions of the Black-Scholes Model

The model assumes:

1. **\*\*The asset price follows a geometric Brownian motion\*\*** with constant drift and volatility, which implies that returns are normally distributed and prices follow a log-normal distribution.
2. **\*\*No arbitrage opportunities\*\*** exist in the market.
3. **\*\*No dividends\*\*** are paid on the underlying asset during the life of the option (though this assumption can be modified in adjusted versions of the model).
4. **\*\*Constant risk-free interest rate\*\*** over the option's life.
5. **\*\*European-style exercise\*\***, meaning the option can only be exercised at expiration.

The Black-Scholes model marked a significant advance in financial mathematics and is foundational in the pricing of derivatives. The use of GBM to model the price path of the underlying asset echoes earlier ideas from Samuelson's work on efficient markets and

random walks, making it a practical application of those theoretical insights.

**my input: What were the breakthrough notions that powered this model's success?**

The Black-Scholes model's success stemmed from several breakthrough notions that not only provided a method for pricing options but also revolutionized modern finance. These core insights include:

1. **No-Arbitrage Principle**: The model fundamentally relies on the no-arbitrage condition, which asserts that there should be no way to make a risk-free profit without any investment. By constructing a risk-free portfolio through dynamic hedging of the option with the underlying asset, Black and Scholes derived an equation that eliminates risk, making option pricing a matter of solving a differential equation under this condition.

2. **Dynamic Hedging**: The Black-Scholes model introduced the concept of continuously adjusting a portfolio by buying or selling the underlying asset to offset changes in the option's value. This idea of "delta hedging" allows an investor to construct a replicating portfolio that mirrors the payoff of the option. Dynamic hedging was groundbreaking because it provided a strategy to manage risk and to maintain a risk-free position, even though the underlying asset price fluctuates.

3. **Risk-Neutral Valuation**: Black, Scholes, and Merton used the concept of risk-neutral valuation, which posits that, for pricing purposes, we can assume that investors are indifferent to risk. This means the expected return of the asset can be replaced with the risk-free rate when valuing the option. This shift simplifies calculations dramatically and allows the pricing formula to be derived without needing an estimate of the actual return expectation of the asset, focusing instead on volatility as the key factor.

4. **Stochastic Calculus Application (Geometric Brownian Motion)**: By modeling the price of the underlying asset as following a geometric Brownian motion (GBM) with a constant drift and volatility, Black and Scholes were able to capture the randomness and potential for exponential growth in asset prices. GBM, which is based on Samuelson's random walk insights, provided a mathematically tractable way to model price paths and ensure that prices remain positive, aligning with real-world behavior.

5. **Partial Differential Equation (PDE) Approach**: The Black-Scholes model translates the option pricing problem into a partial differential equation—the Black-Scholes PDE. Solving this PDE under the boundary conditions of a European option provides the famous closed-form solution for option prices. This mathematical formulation linked financial options with heat equations in physics, revealing an interdisciplinary bridge between finance and physics.

These innovations transformed options trading and financial risk management by providing a systematic, quantitative framework for pricing derivatives. The model not only made it possible to price options consistently but also laid the groundwork for a vast field

of research and practical applications in quantitative finance.

**my input: is it fair to say that they manipulated the equations to the point where they had the same form as a heat diffusions equation, and then leveraged the solution to that?**

Yes, it's fair to say that Black, Scholes, and Merton manipulated the option pricing problem into a form similar to the heat diffusion equation and then leveraged established methods from physics to solve it. Here's how they did it:

1. **\*\*Recasting the Problem as a PDE\*\***: By using stochastic calculus, specifically Itô's lemma, Black and Scholes derived a partial differential equation (PDE) that describes the dynamics of the option price in terms of the underlying asset's price and time. This PDE, now known as the Black-Scholes equation, resembles the heat equation from physics, which describes how heat diffuses through a medium over time.

The Black-Scholes PDE for a European call option  $C$  is:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

This form is strikingly similar to the heat diffusion equation in physics:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

where  $u$  is the temperature,  $D$  is the diffusion coefficient,  $t$  is time, and  $x$  is position.

2. **\*\*Boundary Conditions\*\***: By recognizing this analogy, Black and Scholes applied the boundary conditions specific to an option: at expiration, the value of a European call option should equal the payoff  $\max(S - K, 0)$ , where  $S$  is the asset price and  $K$  the strike price. This is akin to specifying boundary conditions in a heat equation for a specific temperature profile at a given point in time.

3. **\*\*Solution via Analogy to Heat Diffusion\*\***: With the PDE established, they could use techniques from the solution of heat equations to solve for the option price explicitly. By transforming variables, they reduced the PDE to a standard heat equation form, allowing them to apply the solution directly. This transformation is often called the "Feynman-Kac formula" in stochastic processes, but in essence, it's the method physicists use to solve heat equations, repurposed for finance.

4. **\*\*Closed-Form Solution\*\***: Leveraging this approach, Black and Scholes obtained a closed-form solution for European options, which could then be used for pricing without further numerical methods. This approach allowed them to bypass more complex probabilistic interpretations and instead solve the problem using well-established mathematical physics techniques.

In sum, **\*\*they ingeniously translated the option pricing problem into a familiar form from physics and borrowed the solution methods of the heat diffusion equation\*\***—an

insightful mathematical shortcut that opened up new possibilities for financial modeling.

**my input: so the BS PDE doesn't look much like the heat diffusion equation to me. Give more detail on that**

You're correct that the Black-Scholes PDE doesn't immediately look like the standard form of the heat diffusion equation. However, through a series of transformations, we can indeed recast it into a form that resembles the heat equation, allowing us to apply known solutions from physics. Let's go through this process in more detail.

The Black-Scholes PDE and the Heat Equation The Black-Scholes PDE for a European call option  $C(S, t)$  on an underlying asset with price  $S$  and time  $t$  is given by:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

This PDE indeed looks quite different from the classic heat equation:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

where  $u = u(x, t)$  is a function of position  $x$  and time  $t$ , and  $D$  is the diffusion coefficient. However, we can transform the Black-Scholes equation into a form resembling the heat equation by introducing substitutions that simplify the dependencies on  $S$  and  $t$ .

#### Step-by-Step Transformation

To transform the Black-Scholes PDE, let's make a series of variable changes.

1. **\*\*Logarithmic Transformation of Stock Price\*\***: Define a new variable  $x = \ln(S)$ , which transforms the Black-Scholes PDE in terms of  $S$  into one in terms of  $x$ . Since  $S = e^x$ , we can use the chain rule to find derivatives with respect to  $S$  in terms of  $x$ .

2. **\*\*Introduce a New Function\*\***: Define  $V(x, \tau) = C(S, t)$ , where  $\tau = T - t$  (with  $T$  being the option's expiration date). This changes the time variable to a "time-to-expiry" perspective, so that  $\tau$  counts down from  $T$  to 0.

3. **\*\*Substitute and Simplify\*\***: Using these substitutions, the derivatives of  $C(S, t)$  with respect to  $S$  and  $t$  become:

$$-\frac{\partial C}{\partial t} = -\frac{\partial V}{\partial \tau} - \frac{\partial C}{\partial S} = \frac{1}{S} \frac{\partial V}{\partial x} - \frac{\partial^2 C}{\partial S^2} = \frac{1}{S^2} \frac{\partial^2 V}{\partial x^2}$$

Substituting these into the Black-Scholes PDE and simplifying, we get:

$$-\frac{\partial V}{\partial \tau} + \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} - rV = 0$$

4. **\*\*Eliminate the Linear Term\*\***: To make the equation look more like the heat equation, let's define  $V(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau)$ , where  $\alpha$  and  $\beta$  are constants to be chosen to simplify the equation. Substituting this expression into the transformed PDE and choosing  $\alpha = -\frac{r}{\sigma^2} + \frac{1}{2}$  and  $\beta = r$ , we obtain:

$$\frac{\partial u}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2}$$

The Result: A Heat Equation Form

This resulting equation,

$$\frac{\partial u}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2}$$

is now in the form of a heat equation in terms of  $u(x, \tau)$ . Here,  $\frac{1}{2} \sigma^2$  acts as the "diffusion coefficient" for the process, similar to  $D$  in the classic heat equation.

Interpretation and Solution

This transformed equation, now resembling the heat equation, allows us to apply methods from heat diffusion analysis, such as Fourier transforms or Green's functions, to solve for  $u(x, \tau)$ . Once we have  $u(x, \tau)$ , we can reverse the transformations to express  $V(x, \tau)$  and thus recover the original option price  $C(S, t)$ .

Summary By transforming the Black-Scholes PDE through logarithmic and time-to-expiry changes, along with an exponential scaling, we arrive at a form that resembles the heat diffusion equation. This similarity allows us to leverage established methods from physics to solve the PDE, leading to the closed-form Black-Scholes formula for European option prices.

**my input: so the BS PDE, before the transformations to make it look like the heat diffusion equation has already eliminated  $\mu$ , the drift of the underlying security from the equation?**

Yes, that's correct. The Black-Scholes PDE notably **does not contain** the drift term  $\mu$  of the underlying security. This is one of the model's key insights and contributions, achieved by leveraging the concept of **risk-neutral valuation** and **no-arbitrage arguments**.

Why and How  $\mu$  is Eliminated The removal of  $\mu$  occurs because of the construction of a **hedged portfolio** that is designed to be risk-free. Here's the process in brief:

1. **Dynamic Hedging**: Black and Scholes construct a portfolio consisting of a long position in the option (to be priced) and a short position in the underlying asset. By continuously adjusting the proportion of the underlying asset (the "delta" hedge), they create a portfolio whose value changes are driven solely by random price fluctuations in the asset, rather than any expected growth due to  $\mu$ .

2. **No-Arbitrage Condition**: Because this hedged portfolio is risk-free, it must earn the risk-free rate  $r$  rather than the risky rate of return  $\mu$ . This no-arbitrage argument allows them to sidestep the need for  $\mu$  in the valuation of the option since the drift of the asset doesn't impact the risk-neutral valuation framework.

3. **Risk-Neutral Measure**: By transforming to a risk-neutral world where all assets

are expected to grow at the risk-free rate  $r$ , they replace the actual drift  $\mu$  with  $r$  in the valuation process. Under this risk-neutral measure, the expected rate of return on the underlying asset is effectively reset to  $r$ , which is consistent with the idea that, under efficient markets, the drift does not affect the "fair" option price.

The Resulting Black-Scholes PDE After setting up this risk-free portfolio and assuming no-arbitrage, the resulting Black-Scholes PDE:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

no longer includes  $\mu$ , the drift of the underlying security. Instead, the term  $r$  (the risk-free rate) appears, which stems from the risk-neutral framework where the portfolio is expected to grow at the risk-free rate. This equation can then be transformed to a heat diffusion-like equation for further solution.

Key Insight This removal of  $\mu$  is a profound insight because it implies that the specific growth expectations of the underlying asset do not affect the option price. \*\*The option price depends only on volatility  $\sigma$ , the risk-free rate  $r$ , and the time to expiration, rather than on the expected return  $\mu$  of the asset itself.\*\* This insight allows the model to price options independently of investors' subjective expectations about asset returns.

**my input: so they effectively eliminated the need to estimate the expected return of the underlying security?**

Yes, exactly. The Black-Scholes model \*\*eliminates the need to estimate the expected return\*\*  $\mu$  of the underlying security, which is one of the model's most significant breakthroughs.

Here's how they achieved this and why it's so impactful:

1. **Risk-Neutral Valuation**: By constructing a risk-free portfolio through delta hedging, Black and Scholes showed that an option's price could be determined without knowing the underlying asset's expected return. Instead of requiring  $\mu$ , they shifted to a **risk-neutral framework** where all securities are assumed to grow at the risk-free rate  $r$ . Under this assumption, the expected return of the asset is replaced by  $r$  for pricing purposes, aligning the valuation with a world where investors are indifferent to risk.

2. **Focus on Volatility and Risk-Free Rate**: The Black-Scholes model thus relies solely on the **volatility**  $\sigma$  of the asset and the **risk-free interest rate**  $r$ . Since volatility can be observed directly from historical prices or implied from market data, and  $r$  is usually known or can be estimated easily, this approach avoids the notoriously challenging task of estimating  $\mu$ , which is both subjective and hard to measure reliably.

3. **No-Arbitrage Argument**: Through the no-arbitrage principle, Black and Scholes created a mechanism to ensure that the value of the option reflects a **self-consistent pricing model**. This means that the price is determined independently of individual or market-wide expectations of asset returns, as long as no arbitrage opportunities exist. The self-hedging nature of the portfolio ensures that, regardless of  $\mu$ , the option price

derived from the model holds.

Impact of Removing  $\mu$  This elimination of  $\mu$  is revolutionary because it makes the model practical and universally applicable. Traders and investors no longer need to agree on an expected return to use the Black-Scholes formula, which simplifies the pricing process and allows for **a standardized, objective pricing method** that relies only on observable market inputs. This insight is foundational to modern financial theory, underpinning not only option pricing but also broader approaches in derivatives pricing where expected returns are often difficult to agree upon or estimate accurately.

So, by effectively removing the need for  $\mu$ , Black and Scholes provided a way to value options without the uncertainties and subjectivity associated with expected return estimation. This made the Black-Scholes model both theoretically robust and practical for widespread use.

**my input: so what are the units of  $D$  in the heat equation?**

In the heat equation, the diffusion coefficient  $D$  has units that describe how "quickly" heat (or particles, in a diffusion process) spreads through a medium over time. Specifically, the heat equation is often written as:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

where: -  $u = u(x, t)$  represents the temperature (or concentration, in diffusion) as a function of position  $x$  and time  $t$ , -  $D$  is the diffusion coefficient.

To determine the units of  $D$ , we can analyze each term's units to ensure they match.

1. **Units of  $\frac{\partial u}{\partial t}$** : This term represents the rate of change of temperature with respect to time. - If  $u$  has units of temperature ( $^{\circ}\text{C}$  or  $\text{K}$ ), and  $t$  is time (seconds,  $\text{s}$ ), then  $\frac{\partial u}{\partial t}$  has units of temperature per time, or  $\text{K} \cdot \text{s}^{-1}$ .

2. **Units of  $\frac{\partial^2 u}{\partial x^2}$** : This term represents the second spatial derivative of temperature. - If  $x$  is position (meters,  $\text{m}$ ), then  $\frac{\partial^2 u}{\partial x^2}$  has units of temperature per square meter, or  $\text{K} \cdot \text{m}^{-2}$ .

3. **Units of  $D$** : For both sides of the equation to match,  $D$  must have units that, when multiplied by  $\text{K} \cdot \text{m}^{-2}$ , yield  $\text{K} \cdot \text{s}^{-1}$ . Thus, the units of  $D$  must be:

$$D = \frac{\text{m}^2}{\text{s}}$$

Summary The diffusion coefficient  $D$  has units of **square meters per second** ( $\text{m}^2/\text{s}$ ). This represents the rate at which heat (or particles, in diffusion) spreads across a spatial domain over time. In the context of heat conduction,  $D$  can also be called the "thermal diffusivity" and reflects how quickly temperature changes propagate through a material.

**my input: what are the units of variance in the BS PDE?**

In the Black-Scholes PDE, the term involving variance appears as part of the second derivative with respect to the stock price,  $S$ , and is associated with the volatility  $\sigma$  of the asset price. The Black-Scholes PDE is given by:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

where: -  $C$  is the option price, -  $S$  is the asset price, -  $r$  is the risk-free interest rate (units of  $s^{-1}$ ), -  $t$  is time, -  $\sigma$  is the volatility (standard deviation of returns).

In this equation, **the term  $\sigma^2$  represents the variance** of the asset's returns.

Determining the Units of Variance ( $\sigma^2$ )

1. **Volatility  $\sigma$** : Volatility  $\sigma$  represents the standard deviation of the asset's returns over time. If we measure returns per unit time (typically per year in finance), then: -  $\sigma$  has units of **time<sup>-1/2</sup>**, often year<sup>-1/2</sup>.

2. **Variance  $\sigma^2$** : Since variance is the square of volatility, the units of  $\sigma^2$  will be: - **time<sup>-1</sup>**, or year<sup>-1</sup> in annualized terms.

Why These Units Are Consistent in the Black-Scholes PDE

The variance term  $\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}$  must have units of price per unit time, matching the time derivative term  $\frac{\partial C}{\partial t}$  (price per time).

-  $\sigma^2$  has units of time<sup>-1</sup>. -  $S^2$  has units of currency<sup>2</sup> (if the asset price  $S$  is in currency). -  $\frac{\partial^2 C}{\partial S^2}$  has units of currency<sup>-1</sup> (since it's the second derivative of price with respect to price).

Combining these, the entire term  $\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}$  has units of currency  $\cdot$  time<sup>-1</sup>, consistent with the other terms in the equation.

In summary, **the units of variance  $\sigma^2$  in the Black-Scholes PDE are time<sup>-1</sup>** (e.g., year<sup>-1</sup> in financial contexts).

**wow! Impressive stuff**

Thank you! The Black-Scholes model is indeed a masterpiece, blending finance, mathematics, and physics in a beautiful way. It's amazing how the insights from seemingly unrelated fields like heat diffusion can help solve complex financial problems! If you have any more questions or want to dive deeper into any other concepts, I'm here to help!