The Expected Value Equal One Case

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Consider the case of n independent, and identical random (iid) variables X_1, X_2…Xn, where the random variables X*i* are lognormally distributed with parameters u_g and σ_g that satisfy the equation

$$u_g = -\frac{{\sigma_g}^2}{2}.$$

Where

- u_g = geometric mean, the average of ln (X*i*)
- σ_g = geometric standard deviation, the standard deviation of ln (X*i*)

What is the expected value $E[X_1X_2..X_n]$ as n → Infinity?

The Status Quo

The status quo solution is based on the assertion that if random variables X1 through Xn are independent and identically distributed, then the theorem below can be used . It asserts for independent random variables

$E[X_1X_2..Xn] = E[Xi]^n$ Theorem 1 (MIT Random Variables and Expectation Corollary 10.2 page 31)

For a log normal distribution the expected value of E[Xi] = $e^{u_g+\frac{{\sigma_g}^2}{2}}$, substituting $-\frac{{\sigma_g}^2}{2}$ for u_g we have $e^{-\frac{{\sigma_g}^2}{2}+\frac{{\sigma_g}^2}{2}}$ $= e^{0}$ = 1

According to theorem 1, $E[X_1X_2..Xn] = E[Xi]^n$ so $E[X_1X_2..Xn] = 1^n$ and $\lim_{n\to\infty}(1)^n = 1$.

Another Way

However, instead of using theorem 1 we can compute $E[X_1X_2..Xn]$ as n → Infinity directly.

The random variable for a log normal distribution is X*i* = $e^{u_g+\sigma_g Zi}$,

where Z_i = standard normal distribution $\mathcal{N}(0,1)$.

Multiplying the random variables $X_1X_2..X_n$ we have $X_1X_2..X_n = \left(e^{u_g+\sigma_g Z_1}\right)\left(e^{u_g+\sigma_g Z_2}\right)\ldots\left(e^{u_g+\sigma_g Z_n}\right)$.
We can rewrite this as

$e^{nu_g}e^{\,n\sigma_g(Z_1+Z_2+\cdots Z_n)}$ which we can rewrite as

$e^{nu_g}e^{\,n\sigma_g\frac{(Z_1+Z_2+\cdots Z_n)}{n}}$ which equals $e^{nu_g}\left(e^{\,\sigma_g\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)}\right)^n$.

For this case $\frac{{\sigma_g}^2}{2}$ must be positive therefore u_g must be negative, so as n → ∞ the $\lim_{n\to\infty} e^{nu_g} = e^{-\infty}$ which equals zero.

The $\left(e^{\sigma_g\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)}\right)^n$ term includes the expression $\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)$ which by the law of large numbers (LLN) $\frac{(Z_1+Z_2+\cdots Z_n)}{n} \xrightarrow{a.s.}$ E$[Z_i]$ as n → ∞. Since $\lim_{n\to\infty}\left(e^{\sigma_g\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)}\right)^n$ = $\left[\lim_{n\to\infty} e^{\sigma_g\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)}\right]^n$ we can invoke the LLN to say = $\left[\lim_{n\to\infty} e^{\sigma_g\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)}\right]^n$ = $\left[\lim_{n\to\infty} e^{\sigma_g \mathrm{E}[Z_i]}\right]^n$. In this case E$[Z_i] = 0$, So we have $\left[\lim_{n\to\infty} e^0\right]^n = (1)^n = 1.$

So the overall $e^{nu_g}\left(e^{\sigma_g\left(\frac{(Z_1+Z_2+\cdots Z_n)}{n}\right)}\right)^n$ expression, the product of the random variables reduces to $(0)(1)$ = 0 as n → ∞.

So as n → Infinity a direct calculation gives us E[X_1X_2..Xn] = E[0] = 0 for this case, which contradicts the solution based on theorem one.

The Coin Flip Contradiction Case   Vance Harwood   19-Oct-2021

Take a fair coin, label the sides 1.2499 for heads and 0.7999 for tails. The expected value of the coin toss, which we will designate as the random variable X*i*, is the arithmetic average of the sides, $\frac{(1.2499+0.7999)}{2}$, so E[X*i*] = 1.0249. The product of a head/tail pair, $(1.2499)(0.7999)$ is approximately 0.9998. For a large number of tosses, what is the expected value of multiplying together the values of *n* coin tosses?

The Status Quo

Given that random variables X_1 through Xn are independent and identically distributed, the standard solution to this problem is to use Theorem 1 below. It asserts:

$E[X_1X_2..Xn] = E[Xi]^n$ Theorem 1

Since in this case E[X*i*] = 1.0249, then according to Theorem 1 the expected value of the multiplied random variable $E[X_1X_2..Xn] = (1.0249)^n$. Since the $\lim_{n\to\infty}(1.0249)^n = \infty$ then $E[X_1X_2..Xn] = \infty$ also.

The Contradiction

However, consider the following calculation of $E[X_1X_2...Xn]$. There are three catagories to consider as n → infinity:

1. The number of tails is larger than the number of heads
2. The number of heads equals the number of tails.
3. The number of heads is larger than the number of tails

The analysis for the first two categories can be combined and represents approximately half the outcomes of this exercise. If the number of tails in the $X_1X_2...Xn$ product is equal or greater than the number of heads then there are $\frac{n-(T-H)}{2}$ pairs of $(1.2499)(0.7999)$ in the product. Multiplying this term by $(0.799)^{\mathrm{T-H}}$ captures the contribution of any unpaired tails. The resulting equation is:

$X_1X_2...Xn = ((1.2499)(0.7999)\,)^{\frac{n-(T-H)}{2}}(0.7999)^{T-H}$

In this scenario the absolute values of both of the base terms of the equation are positive numbers, less than one and the exponents are non-negative. Therefore, the overall product will always be less than one. In addition, as n → ∞ for y > 0 and y < 1 the $\lim_{n\to\infty}(y)^n = 0$. So, we can say in the limit:

$E[X_1X_2...Xn] = E[\,\lim_{n\to\infty}\left((0.7999)(1.2499^{\frac{n-(T-H)}{2}}(0.7999)^{T-H}\right) = E[0] = 0.$

Since this is the outcome of the coin flip exercise approximately half the time, this result falsifies the status quo position that $E[X_1X_2..Xn] = (1.2499)^n$, where $\lim_{n\to\infty}(1.0249)^n \to \infty$.

The analysis of case 3, when the number of heads is larger than the number of tails also contradicts the status quo is below.

The Coin Flip Contradiction: More Heads Than Tails

The mean or expected value of the number of heads for each toss is 0.5 heads. If the coin toss is repeated n times, then as n → infinity the Law of Large Numbers says that the proportion of heads after n flips will almost surely converge to $\frac{1}{2}$ as n → infinity. While the absolute difference between the number of heads and tails can be large, the ratio of that difference divided by the total number of flips will go to zero as n → infinity, $\lim_{n\to\infty} \frac{|H-T|}{n} = 0$.

In the case of more heads than tails we have $\frac{n-(H-T)}{2}$ pairs of $(1.2499)(0.7999)$ multiplied by $(1.2499)^{\mathrm{H-T}}$ which gives us the following equation:

E[X_1X_2...Xn] = $E[\big((1.2499)(0.7999)\big)^{\frac{n-(H-T)}{2}}(1.2499)^{H-T}]$

Rearranging the terms we have

$$\big((1.2499)(0.7999)\big)^{\frac{n-(H-T)}{2}}(1.2499)^{H-T} = \big((1.2499)(0.7999)\big)^{\frac{n}{2}}\big((1.2499)(0.7999)\big)^{\frac{-(H-T)}{2}}(1.2499)^{H-T}.$$

Taking the terms to the $\frac{n}{n}$ power and rearranging we have:

$$\left(\big((1.2499)(0.7999)\big)^{\frac{1}{2}}\big((1.2499)(0.7999)\big)^{-\frac{1}{2}\left(\frac{(H-T)}{n}\right)}(1.2499)^{\left(\frac{(H-T)}{n}\right)}\right)^{n}.$$

The $\lim_{n\to\infty}\left(\big((1.2499)(0.7999)\big)^{\frac{1}{2}}\big((1.2499)(0.7999)\big)^{-\frac{1}{2}\left(\frac{(H-T)}{n}\right)}(1.2499)^{\left(\frac{(H-T)}{n}\right)}\right)^{n}$ = $\left[\lim_{n\to\infty}\left(\big((1.2499)(0.7999)\big)^{\frac{1}{2}}\big((1.2499)(0.7999)\big)^{-\frac{1}{2}\left(\frac{(H-T)}{n}\right)}(1.2499)^{\left(\frac{(H-T)}{n}\right)}\right)\right]^{n}$.

Since the limit of the absolute difference between the number of heads and tails divided by the total number of flips $\lim_{n\to\infty}\frac{|H-T|}{n} = 0$, then as n → infinity the $\frac{(H-T)}{n}$ terms go to zero giving us $\lim_{n\to\infty}\big((1.2499)(0.7999)\big)^{-\frac{1}{2}\left(\frac{(H-T)}{n}\right)}$ =1 and $\lim_{n\to\infty}(1.2499)^{\left(\frac{(H-T)}{n}\right)}$ = 1 so the overall limit as n → ∞ is $\lim_{n\to\infty}\left(\big((1.2499)(0.7999)\big)^{\frac{1}{2}}(1)(1)\right)$ = $\big((1.2499)(0.7999)\big)^{\frac{1}{2}}$. Since $\big((1.2499)(0.7999)\big)^{\frac{1}{2}}$ is less than one and greater than zero then $\lim_{n\to\infty}\left(\big((1.2499)(0.7999)\big)^{\frac{1}{2}}\right)^{n}$ = 0. Therefore for the heads greater than tails case E[X_1X_2...Xn] = E[$\lim_{n\to\infty}\left(\big((1.2499)(0.7999)\big)^{\frac{1}{2}}\big((1.2499)(0.7999)\big)^{-\frac{1}{2}\left(\frac{(H-T)}{n}\right)}(1.2499)^{\left(\frac{(H-T)}{n}\right)}\right)^{n}$] = E[0] =0 which also contradicts the status quo result of E[X_1X_2..Xn] = $(1.2499)^{n}$, where $\lim_{n\to\infty}(1.0249)^{n}$ → ∞

Two Means Straddling One Case

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The Case

Consider the case of n independent, and identical random (iid) variables X_1, X_2...Xn, where the random variables Xi take on positive real number values of *b* or *c* with 0.5 probability. The characteristics of the values b and c are:

1. *v* is a positive, non-infinite real number
2. *d* is a negative, non-infinite real number
3. *b* $=e^{(d+v)}$ and *c* = $e^{(d-v)}$
4. $e^{d} > \frac{2}{(e^{v}+e^{-v})}$

Example parameters that meet these conditions are *v* = 0.2 and *d* = -0.01.

The net effects of these restrictions are:

1. That *b* multiplied by *c* = $e^{(d+v+d-v)} = e^{(2d)}$
2. The arithmetic mean of *b* and *c* will always be greater than one
3. The geometric mean of *b* and *c* will always be less than one
4. The random variable E[X*i*] has a defined mean and variance

The Status Quo

The expected value of E[X*i*] is the arithmetic mean of X*i*, which is

$\frac{e^{(d+v)}+e^{(d-v)}}{2}$ which can be rewritten as $e^{d}\left(\frac{e^{(v)}+e^{(-v)}}{2}\right)$

The term on the left, e^{d} was defined earlier as being more positive than $\frac{2}{(e^{v}+e^{-v})}$. The term $\frac{2}{(e^{v}+e^{-v})}$ is the reciprocal of the right-hand term $\frac{e^{(v)}+e^{(-v)}}{2}$. A number multiplied by its reciprocal equals one, but since e^{d} is defined to be more positive than $\frac{2}{(e^{v}+e^{-v})}$ the product of e^{d} and $\left(\frac{e^{(v)}+e^{(-v)}}{2}\right)$ must be larger than one. Therefore, the expected value, E[X*i*] must be greater than 1.

Given that, theorem 1 states that in this case, $E[X_1X_2..X_n] = E[Xi]^n$ which goes to ∞ as $n \rightarrow \infty$

The Contradiction

$E[X_1X_2...Xn]$ is equal to $E[e^{\ln(X1X2...Xn)}]$ we can further rewrite this as

$E[e^{(ln(X1)+ln(X2)+\cdots ln(Xn))}]$. We multiply the exponent by $\frac{n}{n}$ which we rewrite as

E$[e^{n(\frac{1}{n}(ln\ (X1)+ln(X2)+\cdots ln\ (Xn)))}]$. Equation (1)

Notice that the term $\frac{1}{n}(ln\ (X1) + ln(X2) + \cdots ln\ (Xn))$ is the sample average of all the ln(X*i*) terms

By the Law of Large Numbers we can say the sample average converges almost surely to the expected value E[ln(X*i*)] as n → ∞.

The natural log of the two values *b* and *c* of X*i* are ln($e^{(d+v)}$) and ln($e^{(d-v)}$)) respectively, which reduces to *d+v* and *d-v*. The expected value then of E[ln(Xi)] is calculated as (d+v)*0.5 plus (d-v)*0.5 = 0.5(d+v+d-v) = 0.5(2d) = d.

Returning to equation 1 we can now say that

E$[e^{n(\frac{1}{n}(\ln\ (X1)+\ln\ (X2)+\cdots\ln\ (Xn)))}]$ = E$[\left(e^{(\frac{1}{n}(\ln(X1)+\ln(X2)+\cdots\ln(Xn))}\right)^n]$ Since $\lim_{n\to\infty}[\left(e^{(\frac{1}{n}(\ln(X1)+\ln(X2)+\cdots\ln(Xn))}\right)^n$ = $\left[\lim_{n\to\infty}\left(e^{(\frac{1}{n}(\ln(X1)+\ln(X2)+\cdots\ln(Xn))}\right)\right]^n$ then by LLN

$\left[\lim_{n\to\infty}\left(e^{(\frac{1}{n}(\ln(X1)+\ln(X2)+\cdots\ln(Xn))}\right)\right]^n \xrightarrow{a.s.} [e^{\mathrm{E}[\ln(\mathrm{Xi})]}]^n$. Since $\mathrm{E}[\ln(\mathrm{X}i)] = d$ we have

$[e^{\mathrm{E}[\ln(\mathrm{X}i)]}]^n = [e^{\mathrm{d}}]^n$ and since *d* is always negative then $e^d < 1$ and the $\lim_{n\to\infty}(e^{\mathrm{d}})^n$ = 0

Therefore in this case E[X_1X_2...Xn] = E[0] = 0 which contradicts theorem 1's assertion that E[X_1X_2..Xn] will go to infinity as n → ∞